TWO-DIMENSIONAL STOKES FLOW OF A VISCOUS FLUID WITH A FREE BOUNDARY UNDER THE EFFECT OF CAPILLARY FORCES

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We consider the flow of an incompressible viscous Newtonian fluid under the effect of surface-tension forces, within a singly connected cylindrical region which has a smooth free boundary. The problem is solved by means of a quasi-stationary Stokes approximation [1], which was first developed in [2], and has been applied to the sintering of porous materials [3], the flow of thin films [4, 5], and geophysical problems [6]. The relaxation of small perturbations of a circular cylinder is analyzed in [7]. An analogy is drawn in [9] between the equations of the theory of elasticity and the equations of hydrodynamics in the Stokes approximation [8]. This method is used in [10-12] to describe plane flow under the effect of capillary forces.

The method we employ here [13] is similar to that of [14, 15]. We express the pressure in the form of an expansion in a complete system of orthonormal harmonic functions; we calculate the velocity field on the boundary; and we show how the behavior of "hole" perturbations varies qualitatively with the depth-to-width ratio of the hole.

1. Consider the quasi-stationary Stokes approximation [1] in the case of the plane flow of an incompressible viscous fluid driven by surface-tension forces, in a bounded region g with a free boundary γ. The equations of motion and continuity and the boundary conditions then have the form

$$\partial_\beta P_{\alpha\beta} = 0; \tag{1.1}$$

$$\partial_\beta v_\beta = 0 \quad (\mathbf{x} \in g); \tag{1.2}$$

$$P_{\alpha\beta} n_\beta = -\sigma n_\alpha \partial_\beta n_\beta \quad (\mathbf{x} \in \gamma); \tag{1.3}$$

where $P_{\alpha\beta} n_\beta = -p\delta_{\alpha\beta} + \mu(\partial_\alpha v_\beta + \partial_\beta v_\alpha)$ is the Newtonian stress tensor, v_α and p are the fluid velocity and pressure, μ and σ are the coefficients of dynamic viscosity and surface tension, which are assumed constant, and n_α is the vector of the outer normal to the boundary. Repeated subscripts indicate summation.

The law of evolution of the boundary is defined by the fact that the normal velocity V of the boundary is equal to the normal component of the fluid velocity, i.e.,

$$V = v_\beta n_\beta, \ \mathbf{x} \in \gamma.$$

If the stress tensor is given in the form

$$P_{\alpha\beta} = -\delta_{\alpha\beta}\partial^2_{\gamma\gamma}\varphi + \partial^2_{\alpha\beta}\varphi \tag{1.4}$$

(φ is the Airy function in the theory of elasticity), Eq. (1.1) is fulfilled identically. The boundary condition (1.3) can be written as

$$D_{\alpha\beta}(\partial_\beta\varphi - \sigma n_\beta) = 0, \ \mathbf{x} \in \gamma \tag{1.5}$$

($D_{\alpha\beta} = n_\alpha\partial_\beta - n_\alpha\partial_\beta$ is the tensor operator of differentiation along the tangent to the boundary, dual to the operator $\mathbf{n} \times \nabla$). Integration of (1.5) yields

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$$\partial_\alpha \varphi = \sigma n_\alpha, \ \mathbf{x} \in \gamma. \tag{1.6}$$

The additive constant which commonly appears in the right-hand side of (1.6) can be eliminated by substituting $\varphi \to \varphi + C_\beta x_\beta$, , which does not alter (1.4).

From (1.4) we have

$$\partial^2_{\beta\beta}\varphi = 2p. \tag{1.7}$$

According to (1.1) and (1.2), p is a harmonic function, and consequently φ is biharmonic.

2. Consider an arbitrary function Ξ which is harmonic in g. Taking the scalar product and vector product of (1.6) and $n_\alpha \Xi$, and then integrating over the boundary γ using (1.7), we obtain

$$\langle p\Xi\rangle_g = \frac{\sigma l}{2S}\langle \Xi\rangle_\gamma; \tag{2.1}$$

$$\langle \nabla\varphi \times \nabla\Xi\rangle_g = 0, \tag{2.2}$$

where $\langle f\rangle_g = S^{-1}\int f dS$, $\langle f\rangle_\gamma = l^{-1}\int f dl$ are the mean values over the region and its boundary, and S and ℓ are the area and perimeter of the region. When $\Xi = 1$ (2.1) becomes $\langle p\rangle_g = \sigma l/(2S)$. This result can be found in [8].

Setting $\Xi = x_1^2 - x_2^2$ in (2.1) yields

$$\langle x_1\partial_2\varphi - x_2\partial_1\varphi\rangle_g = 0. \tag{2.3}$$

We further have from (1.6)

$$\langle \nabla\varphi\rangle_g = 0. \tag{2.4}$$

Introducing in g a complete system of orthonormal harmonic functions $\{\Xi_k\}_k^\infty$ which obey the orthogonality condition [16] $\langle \Xi_k \Xi_n\rangle_g = \delta_{kn}$, we obtain from (2.1) the following expression for the pressure:

$$p = \langle p\rangle_g \sum_{k=0}^{\infty} \Xi_k \langle \Xi_k\rangle_\gamma. \tag{2.5}$$

The series converges because the system of functions by applying the Gram–Schmidt orthogonalization process [17] to a complete but non-orthogonal system of real functions harmonic in g; thus

$$\{\chi_k\}_{k=0}^{\infty} = \{1, x_1, x_2; \ \ldots; \operatorname{Re} z^k, \operatorname{Im} z^k; \ \ldots\}, \quad z = x_1 + ix_2.$$

Then

$$\Xi_k = \frac{1}{\sqrt{\Delta_k \Delta_{k-1}}}\begin{vmatrix} a_{11} & \ldots & a_{1,k-1} & \chi_1 - \langle\chi_1\rangle_g \\ \cdot & \cdot & \cdot & \cdot \\ a_{11} & \ldots & a_{k,k-1} & \chi_k - \langle\chi_k\rangle_g \end{vmatrix}, \tag{2.6}$$

where

$$\Delta_k = \begin{vmatrix} a_{11} & \ldots & a_{11} \\ \cdot & \cdot & \cdot \\ a_{11} & \ldots & a_{11} \end{vmatrix}; \ a_{km} = \langle \chi_k \chi_m\rangle_g - \langle\chi_k\rangle_g \langle\chi_m\rangle_g.$$

Inserting (2.6) in (2.5) yields

$$p = \langle p \rangle_g \left(1 + \sum_{k=1}^{\infty} c_k \Xi_k\right)$$

$$\left(c_k = \frac{1}{\sqrt{\Delta_k \Delta_{k-1}}} \begin{vmatrix} a_{11} & \cdots & a_{1,k-1} & \langle \chi_1 \rangle_\gamma - \langle \chi_1 \rangle_g \\ \cdot & \cdot & \cdot & \cdot \\ a_{11} & \cdots & a_{k,k-1} & \langle \chi_k \rangle_\gamma - \langle \chi_k \rangle_g \end{vmatrix}\right). \tag{2.7}$$

From [19] we have

$$v_\alpha = \frac{1}{2\mu}(\partial_\alpha \varphi - \Phi_\alpha), \quad \mathbf{x} \in \gamma, \tag{2.8}$$

where

$$(\Phi_1, \Phi_2) = (\mathrm{Re}\, \Phi, \ \mathrm{Im}\, \Phi), \ d\Phi = \Pi dz, \ \Pi = p + i\omega,$$
$$\omega = \mu(\partial_2 v_1 - \partial_1 v_2). \tag{2.9}$$

The function Π is analytic, since the Cauchy–Riemann conditions $\partial_1 \omega = -\partial_2 p, \ \partial_2 \omega = \partial_1 p$ are fulfilled identically by the equations of motion. If we reduce this complex function to its real part and find the function Φ from (2.9), we obtain

$$\Phi = \langle p \rangle_g \left(z + \sum_{k=1}^{\infty} c_k \begin{vmatrix} a_{11} & \ldots & a_{1,k-1} & A_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{11} & \ldots & a_{k,k-1} & A_k \end{vmatrix}\right) + \Phi_0 + i\omega_0 z. \tag{2.10}$$

Here

$$A_n = \frac{\varkappa_{n+1}}{[(n+1)/2]+1} - z\langle \chi_{n+1} \rangle_g; \quad \varkappa_{2n-1} = z^k; \quad \varkappa_{2n} = -iz^k;$$

Φ_0 and ω_0 are arbitrary constants (the first is complex and the second is real). In order to determine Φ_0, we require that the momentum of the system should be zero, i.e., $\langle v_\alpha \rangle_g = 0$. Then, on the basis of (2.4) and (2.8),

$$\langle \Phi \rangle_g = 0. \tag{2.11}$$

We assume that the origin coincides with the center of inertia, i.e.,

$$\langle x_\alpha \rangle_g = 0. \tag{2.12}$$

Since $d\langle x_\alpha \rangle_g/dt = \langle v_\alpha \rangle_g = 0$, if (2.12) holds initially, it will also continue to hold subsequently.

To determine ω_0, we require that the angular momentum of the system should be zero, i.e., $\langle v_1 x_2 - v_2 x_1 \rangle_g = 0$. We then obtain from (2.3) and (2.8)

$$\mathrm{Im}\, \langle \Phi z \rangle_g = 0. \tag{2.13}$$

Using (2.11) and (2.13) to find the constants Φ_0 and ω_0, we finally obtain for (2.10)

$$\Phi = \langle p \rangle_g \left(z + \sum_{k=1}^{\infty} c_k \begin{vmatrix} a_{11} & \ldots & a_{1,k-1} B_1 \\ \cdot & \cdot & \cdot \\ a_{11} & \ldots & a_{k,k-1} B_k \end{vmatrix}\right), \tag{2.14}$$

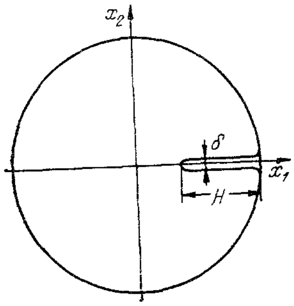

Fig. 1

where

$$B_n = ([(n+1)/2]+1)^{-1}\langle \varkappa_{n+1} - \langle \chi_{n+1}\rangle_g - -iz\,\mathrm{Im}\,\langle \varkappa_{n+1}\bar{z}\rangle_g / /\langle z\bar{z}\rangle_g) - i\langle \chi_n\rangle_g z.$$

Separating real-valued and imaginary parts of Φ and utilizing (1.6), (2.8), we obtain the distribution velocity at the boundary

$$v_\alpha = \frac{1}{2\mu}(\sigma n_\alpha - \Phi_\alpha), \quad \mathbf{x} \in \gamma.$$

The normal velocity of the boundary defines its law of evolution, i.e.,

$$V = \frac{1}{2\mu}(\sigma - n_\beta \Phi_\beta), \quad \mathbf{x} \in \gamma. \tag{2.15}$$

3. Consider a small perturbation of the circular cylinder boundary, given by $r = R + h(\varphi, t)$; we then have from (2.14) and (2.15)

$$\frac{\partial h}{\partial t} = -\frac{\sigma}{2\mu R}\sum_{k=-\infty}^{\infty} |k| \exp(ik\varphi)\, h_k, \quad h_k = \int_0^{2\pi} \frac{d\varphi}{2\pi}\exp(-ik\varphi), \tag{3.1}$$

in agreement with [7]. According to (3.1), a small boundary perturbation of characteristic width $\delta \ll R$ and amplitude $H \ll \delta$ has a characteristic decay time $\tau \sim \mu\delta/\sigma$.

Let now consider a large amplitude perturbation in the shape of "hole" (Fig. 1). By symmetry, the pressure must be an even function with respect to x_2, i.e., $p(x_1, -x_2) = p(x_1, x_2)$. We introduce a space of two-variable harmonic functions which are even with respect to the second argument, and choose in it a complete system of functions of the form $\chi_n = r^n \cos(n\varphi)$ (r and φ are polar coordinates in the x_1, x_2 plane). Since the width δ is small, $\langle \chi_m \chi_n\rangle_g = R^{2n}\delta_{mn}/2(n+1)$. Then the complete system of orthonormal harmonic functions in that space is

$$\Xi_n = \sqrt{2(n+1)}(r/R)^n \cos(n\varphi). \tag{3.2}$$

Inserting (3.2) in (2.5) and summing the series yields

$$p = \sigma\left[\frac{1}{R} - \frac{H}{\pi R^2} - \frac{2}{\pi}\mathrm{Re}\left(\frac{1}{R-z} - \frac{R-H}{R^2-(R-H)z}\right)\right], \tag{3.3}$$

whence, using (2.12), we have

$$\Phi = \sigma\left[\left(1 - \frac{H}{\pi R}\right)\frac{z}{R} + \frac{2}{\pi}\ln\frac{R^2-(R-H)z}{(R-H)R}\right]. \tag{3.4}$$

In spite of the logarithm, (3.4) is a single-valued analytic function in g, because the boundary perturbation constitutes a branch cut. If we insert (3.4) in (2.16), we find that the normal velocity of the cut edges $V = \sigma/(2\mu)$ (in the zero approximation with respect to the small parameter δ/H). The edges close up after a time $\tau = \mu\delta/\sigma$. Although capillary forces generally tend to flatten the boundary perturbations, in this case they produce the opposite effect. In acting to reduce the length of the cut, the capillary forces generate a flow of scale H in the region. The velocities along x_1 and x_2 have the scales $\dot{H}$ and $\dot{\delta}$, respectively. If we equate the work of surface-tension forces with the rate of energy dissipation by viscous forces, we find that $\sigma\dot{H} \simeq -\mu(\dot{H}/H)^2H^2$ or $\dot{H} \simeq \dot{\delta} \simeq -\sigma/\mu$; this conforms to the rigorous result we obtained before.

At the time limit $\delta \to 0$, the boundary curvature at the point $x_1 = R - H$, $x_2 = 0$ becomes infinite. According to (3.3), however, this does not give rise to a singularity in the pressure field at the point. As in the case of a collapsing circular capillary [20], both the capillary and the viscous forces increase with the curvature; they counteract each other, and the pressure thus remains finite.

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